Simple Neuron Models: FitzHugh-Nagumo and Hindmarsh-Rose

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- Reduction of the Hodgkin-Huxley model
- The FitzHugh-Nagumo model
- Phase plane analysis
- Excitability (threshold-like behavior), periodic spiking (Hopf bifurcation)
- The Hindmarsh-Rose model for bursting neurons
Neuron models (sketch)

Single Neurons

experiments

Hodgkin–Huxley, 1952
− current based
reduction
detailed, specific models
− compartmental (structure)
− more currents
− adaptive (state–dep. prop.)
low–dimensional models
− FitzHugh–Nagumo, 1960’s
− Hindmarsh–Rose, 1980’s

Networks

simplification
* effective numerical simulation
* allow for most common features
− excitability
− spiking, different time scales
integrate–and–fire models
stochastic models
abstraction
Hopfield network, 1980’s
− on–off neuron, learning, stat. physics
Hodgkin-Huxley model

- neuronal signals are short electrical pulses: spikes or action potentials on msec scale
- intracellular: incoming spike modifies membran potential

Hodgkin-Huxley (1952): Semirealistic 4-dimensional model for the dynamics of the membran potential by taking into account Na+, K+, and a leak current. Dynamics of ion channels highly nonlinear ⇒ emergence of chaotic evolution.

membran potential: \[\frac{dV}{dt} = C_Na m^3 h (E_Na - V) + C_K n^4 (E_K - V) + C_{\text{leak}} (V_{\text{rest}} - V) + I_{\text{inj}}(t)\]

sodium \(I_{Na}\), fast:
\[\frac{dm}{dt} = \alpha_m(V) (1 - m) - \beta_m(V) m\]

slow:
\[\frac{dh}{dt} = \alpha_h(V) (1 - h) - \beta_h(V) h\]

potassium \(I_{K}\), slow:
\[\frac{dn}{dt} = \alpha_n(V) (1 - n) - \beta_n(V) n\]
Dynamics of currents $m, h, n$

General form:

$$\frac{dx}{dt} = -\frac{1}{\tau(V)}[x - x_s(V)]$$

Solution for constant $V$:

$x(t) = (x_0 - x_s) \exp(-t/\tau) + x_s$

$\Rightarrow$ exponential relaxation to steady state value $x_s$

For varying $V(t)$:

$x(t)$ follows varying steady state value $x_s(t)$

small $\tau$: fast relaxation $\Rightarrow$ $x(t) \approx x_s(t)$

large $\tau$: slow dynamics
Reduction to two-dimensional model

fast sodium dynamics:

approximate by steady state value: \( m(t) \approx m_s(V) \)

similar dynamics of slow sodium and potassium:

replace \( h(t), n(t) \) by one effective current \( w(t) \)

\[ \Rightarrow \] two equations for temporal evolution of \( V(t) \) and \( w(t) \)
FitzHugh-Nagumo model

FitzHugh (1961) and Nagumo, Arimoto, Yoshizawa (1962) derived 2-dimensional model for an **excitable** neuron:

- **membrane potential:**\[ \frac{dv}{dt} = v - \frac{v^3}{3} - w + I \]
- **current variable:**\[ \frac{dw}{dt} = \frac{1}{\tau}(v + a - bw) \]

**typical values:** \( a = 0.7, b = 0.8, \tau = 13 \)

\[ \Rightarrow \frac{\dot{v}}{\dot{w}} \sim 10 \Rightarrow w \text{ slow}, \ v \text{ fast} \]

For constant input \( I = \text{const} \) no chaotic evolution
Phase plane analysis

Two-dimensional flow field:

\[ \vec{F}(v, w) = \frac{d}{dt} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} v - \frac{v^3}{3} - w + I \\ \frac{1}{\tau}(v + a - bw) \end{pmatrix} \]

(numerical) solution: \[ \begin{pmatrix} v(t) \\ w(t) \end{pmatrix} \Rightarrow \text{trajectory in 2-D plane} \]

Characteristics:

- trajectories cannot cross (uniqueness of solutions)
- **nullclines** define lines in the 2-D plane:
  \[ \begin{align*}
  \dot{v} &= 0 \quad \Rightarrow \quad w = v - \frac{v^3}{3} + I \\
  \dot{w} &= 0 \quad \Rightarrow \quad w = \frac{v + a}{b}
  \end{align*} \]
- crossings of the nullclines correspond to fixed points (stable for \( I = 0 \))
Phase plane portrait of FitzHugh-Nagumo model for $I = 0$

arrows indicate flow field $(\dot{v}, \dot{w})$
Subthreshold pulse injection

injection of weak pulse $I(t) = I_0 \delta(t - t_0)$: fast return to FP
Subthreshold pulse injection

no action potential
Suprathreshold pulse injection

Stronger pulses: large excursion in phase plane
Suprathreshold pulse injection

spike response – action potential generation
Refractory period

Immediately after spike the neuron is indifferent to further input
FitzHugh-Nagumo model for constant $I > 0$

Phase plane analysis:
$I$ shifts nullcline of $v$, nullcline of $w$ unaffected

$$\dot{v} = 0 : \quad w = v - \frac{v^3}{3} + I,$$
$$\dot{w} = 0 : \quad w = \frac{(v+a)}{b}$$

$\Rightarrow$ for large enough $I > 0.33$ the fixed point, $\dot{v} = \dot{w} = 0$, becomes unstable

$\Rightarrow$ Onset of **sustained oscillations** (Hopf-bifurcation)
Nullclines for constant $I > 0$

$v$ - nullcline shifted $\Rightarrow$ for $I > 0.33$ the fixed point becomes unstable
Below the bifurcation, $I = 0.3$

Fixed point remains stable $\Rightarrow$ small damped oscillations
Below the bifurcation, $I = 0.3$

Fixed point remains stable $\Rightarrow$ small damped oscillations
Above the bifurcation, $I = 0.4$

Fixed point unstable $\Rightarrow$ Hopf-bifurcation to sustained oscillations on limit cycle
Above the bifurcation, $I = 0.4$

Fixed point unstable $\Rightarrow$ periodic spiking
FitzHugh-Nagumo model for varying $I(t)$

Recapitulation:

- For $I = \text{const} > 0.33$ onset of stable oscillations with Frequency $\Omega(I)$
- Refractory period where system is rather indifferent to external signals

Time dependent input:

- periodic signals: resonance effects
- noisy signals: **coherence resonance**
Summary FitzHugh-Nagumo

- two dimensional model that can be derived from Hodgkin-Huxley via reduction of variables
- allows effective phase plane analysis
- excitable: spike response to suprathreshold input pulse
- refractory period
- with increasing input current Hopf-bifurcation to sustained periodic spiking

- reduction of complexity: no self-sustained chaotic dynamics
- no bursting
- few parameters: difficult to adapt to neurons with specific properties
The Hindmarsh-Rose model

Developed 1982-1984 by J. L. Hindmarsh and R. M. Rose to allow for rapid firing or bursting

Idea:
Allow for triggered firing, i.e., switch between a stable rest state and a stable limit cycle (rapid periodic firing)
⇒ more than one fixed points required: can be achieved by deformation of the nullclines (nonlinear “current” equation)

Basic equations:
\[
\frac{dx}{dt} = 3x^2 - x^3 - y + I, \quad \frac{dy}{dt} = 5x^2 - 1 - y
\]

Nullclines:
\[
\dot{x} = 0 : \quad y = 3x^2 - x^3 + I, \quad \dot{y} = 0 : \quad y = 5x^2 - 1
\]
Phase portrait of Hindmarsh-Rose model

3 Fixed points $\Rightarrow$ coexistence of rest state and limit cycle

\[
\begin{align*}
\frac{dx}{dt} &= 0 \\
\frac{dy}{dt} &= 0
\end{align*}
\]
Adaption variable

Termination of firing via additional adaption variable $z$ that should:

- lower the effective current when neuron is firing
- return to zero when $x$ has reached its rest state value $x_r$

Complete equations:

$$
\frac{dx}{dt} = 3x^2 - x^3 - y + I - z, \quad \frac{dy}{dt} = 5x^2 - 1 - y, \quad \frac{dz}{dt} = r [s (x - x_r) - z]
$$
Bursting of Hindmarsh-Rose model

After repeated firing the dynamics returns to the stable fixed point
Bursting of Hindmarsh-Rose model

Several spikes with varying interspike-interval (ISI)
Features of the Hindmarsh-Rose model

3-D model for neuron with rapid firing

Suitable choice of parameters allows for

- regular bursting
- chaotic bursting

Suitable choice of parameters ⇔ ? real neurons
Further reading

- W. Gerstner and W. M. Kistler, Spiking Neuron Models: Single Neurons, Populations, Plasticity,
- J. Hindmarsh and P. Cornelius, The Development of the Hindmarsh-Rose model for bursting,