Hysteretic Transitions in the Kuramoto Model with Inertia

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Pteroptix Malaccae

Usually, entrainment results in a constant phase angle equal to the difference between pacing frequency and free-running period as it does in P. cribellata. The mechanism of attaining synchrony by Malaysian firefly Pteroptyx malaccae is quite different. When the pacer changes, this firefly requires several cycles to reach a steady state. Once this steady state is achieved, the phase angle difference is near zero irrespective of the pacer period. This can be explained only by the animal adjusting the period of its oscillator to equal that of the driving oscillator. (experiments by Hanson, 1987)

A phase model with inertia allows for adaptation of its frequency to the forcing one
Plan of the Talk

- Introduction of the Kuramoto model with inertia
- Analogy with the damped oscillator (coexistence of stable periodic and fixed point solutions)
- Mean field theory of the hysteretic transition (Tanaka, Lichtenberg, Oishi 1997)
- Fully coupled network of N oscillators
  - Existence of clusters of locked oscillators of any size between the hysteretic curves
  - Limits of stability of the coherent and incoherent solutions (dependence on the size N and on the mass m)
  - Emergence of drifting clusters
- Diluted network
- Italian high voltage power grid
The Model

Kuramoto model with inertia

\[ m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i + \frac{K}{N} \sum_j \sin(\theta_j - \theta_i) \]

- \( \theta_i \) is the instantaneous phase
- \( \Omega_i \) is the natural frequency of the \( i \)-th oscillator with Gaussian distribution
- \( K \) is the coupling constant
- \( N \) is the number of oscillators

By introducing the complex order parameter

\[ r(t)e^{i\phi(t)} = \frac{1}{N} \sum_j e^{i\theta_j} \]

\[ m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr \sin(\theta_i - \phi) \]

\( r = 0 \) asynchronous state, \( r = 1 \) synchronized state
Damped Driven Pendulum

\[ m\ddot{\theta}_i + \dot{\theta}_i = \Omega_i - Kr \sin(\theta_i) \]

\[ I = \frac{\Omega_i}{Kr} \]

\[ \beta = \frac{1}{\sqrt{mKr}} \]

\[ \ddot{\phi} + \beta \dot{\phi} = I - \sin(\phi) \]

For sufficiently large \( m \) (small \( \beta \))

- For small \( \Omega_i \) two fixed points are present: a \textit{stable node} and a \textit{saddle}.
- At larger frequencies \( \Omega_i > \Omega_P = \frac{4}{\pi} \sqrt{\frac{Kr}{m}} \) a \textit{limit cycle} emerges from the saddle via a homoclinic bifurcation.
- Limit cycle and fixed point coexists until \( \Omega_i \equiv \Omega_D = Kr \), where a saddle node bifurcation leads to the \textit{disappearance of the two fixed points}.
- For \( \Omega_i > \Omega_D \) only the \textit{oscillating solution} is present.

For small mass (large \( \beta \)), there is no more coexistence. (Levi et al. 1978)
Dynamics of \( N \) oscillators

- \( \Omega_M \): maximal natural frequency of the locked oscillators
- \( \Omega_P^{(I)} = \frac{4}{\pi} \sqrt{\frac{K r}{m}} \)
- \( \Omega_D^{(II)} = K r \)

**Protocol I: Increasing \( K \)**

The system remains desynchronized until \( K = K^1_c \) (filled black circles). \( \Omega_M \) increases with \( K \) following \( \Omega_P^{I} \). \( \Omega_i \) are grouped in small clusters (plateaus).

**Protocol II: Decreasing \( K \)**

The system remains synchronized until \( K = K^2_c \) (empty black circles). \( \Omega_M \) remains stucked to the same value for a large \( K \) interval than it rapidly decreases to 0 following \( \Omega_D^{II} \).

\[
m = 2
\]
Tanaka et al. (TLO) in [PRL, Physica D (1997)] examined the origin of the hysteretic transition finding that

- by following Protocol I and II there is a group of drifting oscillators and one of locked oscillators which act separately
  - locked oscillators are characterized by $< \dot{\theta} > = 0$
  - drifting oscillators $< \dot{\theta} > \neq 0$

- Drifting and locked oscillators are separated by a frequency:
  - Following Protocol I the oscillators with $\Omega_i < \Omega_P$ are locked
  - Following Protocol II the oscillators with $\Omega_i < \Omega_D$ are locked

- These two groups contribute differently to the total level of synchronization in the system

$$r = r_L + r_D$$
Total level of synchronization in the system: \[ r = r_L + r_D \]

For the **locked** population TLO derived the self-consistent equation

\[ r_{L}^{I,II} = Kr \int_{-\theta_{P,D}}^{\theta_{P,D}} \cos^2 \theta g(Kr \sin \theta) d\theta \]

where \( \theta_P = \sin^{-1}\left(\frac{\Omega_P}{Kr}\right) \), \( \theta_D = \sin^{-1}\left(\frac{\Omega_D}{Kr}\right) = \pi/2 \). 

For the **drifting** population the self-consistent equation is

\[ r_{D}^{I,II} \simeq -mKr \int_{-\Omega_{P,D}}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega \]

The former equation are correct in the limit of sufficiently large masses.
Hysteretic Behavior

Numerical Results for Fully Coupled Networks ($N = 500$)

- The data obtained by following protocol II are quite well reproduced by the mean field approximation $r^{II}$.
- The mean field estimation $r^I$ by TLO does not reproduce the stepwise structure in protocol I.
- Clusters of $N_L$ locked oscillators of any size remain stable between $r^I$ and $r^{II}$.
- The level of synchronization of these clusters can be theoretically obtained by generalizing the theory of TLO to protocols where $\Omega_M$ remains constant,

$$m = 6$$
Finite Size Effects

- $K_1^c$ is the transition value from asynchronous to synchronous state (following Protocol I)
- $K_2^c$ is the transition value from synchronous to asynchronous state (following Protocol II)
- $K_1^c$ is strongly influenced by the size of the system
- $K_2^c$ does not depend heavily on $N$

Dashed line $\rightarrow K_1^{MF}$ mean field value by Gupta et al (PRE 2014)
The mean field critical value has been estimated Gupta, Campa, Ruffo (PRE 2014) by employing a nonlinear Fokker-Planck formulation for the evolution of the single oscillator distribution $\rho(\theta, \dot{\theta}, \Omega, t)$ for coupled oscillators with inertia and noise

$$\frac{1}{K_{1}^{MF}} = \frac{\pi g(0)}{2} - \frac{m}{2} \int_{-\infty}^{\infty} \frac{g(\Omega)d\Omega}{1 + m^2\Omega^2}$$

where $g(\Omega)$ is an unimodal distribution Acebron et al, PRE (2000)

We observe the following scaling with the system size $N$ for fixed mass

$$K_{1}^{MF} - K_{1}^{C}(N) \propto N^{-1/5}$$

this is true for sufficiently low masses
Dependence On the Mass $K_1^c$

- $K_1^c$ increases with $m$ up to a maximal value and then decreases at larger masses.
- By increasing $N$, $K_1^c$ increases and the position of the maximum shifts to larger masses (finite size effects).

The following general scaling seems to apply:

$$\xi \equiv \frac{K_1^{MF} - K_1^c(m, N)}{K_1^{MF}} = G \left( \frac{m}{N^{1/5}} \right)$$

where $K_1^{MF} \propto 2m$ for $m > 1$.
The TLO approach fails to reproduce the critical coupling for the transition from asynchronous to synchronous state (i.e., $K_1^c$), however it gives a good estimate of the return curve obtained with protocol II from the synchronized to the asynchronous regime.

- $K_2^c$ initially decreases with $m$ then saturates, limited variations with the size $N$
- $K_2^{TLO}$ is the minimal coupling associated to a partially synchronized state given by TLO approach for protocol II
- $K_2^{TLO}$ exhibits the same behaviour as $K_2^c$, however it slightly underestimates the asymptotic value (see the scale)
For larger masses \((m=6)\), the synchronization transition becomes more complex, it occurs via the emergence of clusters of drifting oscillators.

The partially synchronized state is characterized by the coexistence of

- a cluster of locked oscillators with \(<\dot{\vartheta}>\simeq 0\)
- clusters composed by drifting oscillators with finite average velocities

The effect of these extra clusters is to induce (periodic or quasi-periodic) oscillations in the temporal evolution of the order parameter.
If we compare the evolution of the instantaneous velocities $\dot{\theta}_i$ for 3 oscillators and $r(t)$ we observe that

- the phase velocities of $O_2$ and $O_3$ display **synchronized motion**
- the phase velocity of $O_1$ oscillates **irregularly** around zero
- the almost periodic oscillations of $r(t)$ are **driven** by the periodic oscillations of $O_2$ and $O_3$
Drifting Clusters III

- The amplitude of the oscillations of $r(t)$ and the number of oscillators in the drifting clusters $N_{DC}$ correlates in a linear manner.

- The oscillations observable in the order parameter are induced by the presence of large secondary clusters characterized by finite whirling velocities.

- At smaller masses oscillations in $r(t)$ are present, but reduced in amplitude. These oscillations are due to finite size effects since no clusters of drifting oscillators are observed.

- Blue dashed line $\implies$ estimated mean field value $r^f$ by TLO.

- The mean field theory captures the average increase of the order parameter but it does not foresee the oscillations.
**Constraint 1**: the random matrix is symmetric

**Constraint 2**: the in-degree is constant and equal to $N_c$

Diluted or fully coupled systems (whenever the coupling is properly rescaled with the in-degree) display the same phase-diagram.

For very small connectivities the transition from hysteretic becomes continuous.

By increasing the system size the transition will stay hysteretic for extremely small percentages of connected (incoming) links.
The TLO mean field theory still gives reasonable results (70% of broken links).

All the states between the synchronization curves obtained following Protocol I and II are reachable and stable.

These states, located in the region between the synchronization curves, are characterized by a frozen cluster structure, composed by a constant $N_L$.

The generalized mean-field solution $r^0(K, \Omega_0)$ is able to well reproduce the numerically obtained paths connecting the synchronization curves (I) and (II).
Each node is described by the phase:

\[ \phi_i(t) = \omega_{AC} t + \theta_i(t) \]

where \( \omega_{AC} = 2\pi \times 50 \text{ Hz} \) is the standard AC frequency and \( \theta_i \) is the phase deviation from \( \omega_{AC} \).

Consumers and generators can be distinguished by the sign of parameter \( P_i \):

\[ P_i > 0 \ (P_i < 0) \]

corresponds to generated (consumed) power.

\[ \ddot{\theta}_i = \alpha \left[ -\dot{\theta}_i + P_i + K \sum_{ij} C_{i,j} \sin(\theta_j - \theta_i) \right] \]

Average connectivity \( < N_c >= 2.865 \)

[ Filatrella et al., The European Physical Journal B (2008)]
We do not observe any hysteretic behavior or multistability down to $K = 9$.

For smaller coupling an intricate behavior is observable depending on initial conditions.

Generators and consumers compete in order to oscillate at different frequencies.

The local architecture favours a splitting based on the proximity of the oscillators.

Several small whirling clusters appear characterized by different phase velocities.

The irregular oscillations in $r(t)$ reflect quasi-periodic motions.
Main Results

We have studied the synchronization transition for a globally coupled Kuramoto model with inertia for different system sizes and inertia values.

- The transition is hysteretic for sufficiently large masses.
- Clusters of locked oscillators of any size coexist within the hysteretic region.
- A generalization of TLO theory is capable to reproduce all the possible synchronization/desynchronization hysteretic loops.
- The presence of clusters composed by drifting oscillators induces oscillatory behaviour in the order parameter.

The properties of the hysteretic transition have been examined also for random diluted network.

- The main properties of the transition are not affected by the dilution.
- The transition appears to become continuous only when the number of links per node becomes of the order of few units.

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Extension of the Mean Field Theory

In principle one could fix the discriminating frequency to some arbitrary value \( \Omega_0 \) and solve self-consistently

\[
 r = r_L + r_D
\]

\[
 r_{L,II}^I = K r \int_{-\theta_0}^{\theta_0} \cos^2 \theta g(K r \sin \theta) d\theta \quad r_{D,II}^I \simeq -m K r \int_{-\Omega_0}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega
\]

This amounts to obtain a solution \( r^0 = r^0(K, \Omega_0) \) by solving

\[
 \int_{-\theta_0}^{\theta_0} \cos^2 \theta g(K r^0 \sin \theta) d\theta - m \int_{-\Omega_0}^{\infty} \frac{1}{(m\Omega)^3} g(\Omega) d\Omega = \frac{1}{K}
\]

with \( \theta_0 = \sin^{-1}(\Omega_0/K r^0) \). The solution exists if \( \Omega_0 < \Omega_D = K r^0 \).

\( \Rightarrow \) A portion of the \( (K, r) \) plane delimited by the curve \( r^{II}(K) \) is filled with the curves \( r^0(K) \) obtained for different \( \Omega_0 \) values.
Hysteretic Behavior

Fully Coupled Networks

- A *step-wise structure* emerges at larger masses due to the breakdown of the independence of the whirling oscillators.
- The number of locked oscillators $N_L$ follows the same step-wise structure.
- $N_L$ remains constant until it reaches the descending curve.

![Graphs showing step-wise structure and number of locked oscillators $N_L$](image-url)
By following Protocol II

- the system stays in one cluster up to $K = 7$
- at $K = 6$ wide oscillations emerge in $r(t)$ due to the locked clusters that have been splitted in two (is this also the origin for the emergent multistability?)
- By lowering further $K$ several whirling small clusters appear and $r$ becomes irregular