Characterizing the Response of Chaotic Systems

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(Received 9 July 2009; revised manuscript received 25 March 2010)

1 We characterize the response of a chaotic system by investigating ensembles of, rather than single, trajectories. Time-periodic stimulations are experimentally and numerically investigated. This approach allows detecting and characterizing a broad class of coherent phenomena that go beyond generalized and phase synchronization. In particular, we find that a large average response is not necessarily related to the presence of standard forms of synchronization. Moreover, we study the stability of the response, by introducing an effective method to determine the largest nonzero eigenvalue $-\gamma_1$ of the corresponding Liouville-type operator, without the need of directly simulating it. The exponent γ_1 is a dynamical invariant, which complements the standard characterization provided by the Lyapunov exponents.

DOI: PACS numbers: 05.45.Xt, 05.45.Jn, 87.18.Sn

Deterministic chaos is a ubiquitous property of nonlinear dynamical systems that is typically accompanied by an exponential loss of memory. Nonetheless, ensembles of nonlinear elements are manifestly able to efficiently process information, neuronal networks being the most striking such example. Is this because such systems manage to avoid chaos, or there exist mechanisms which guarantee a reliable information processing even in the presence of chaos? The nontrivial collective behavior spontaneously emerging in populations of identical units [1] indeed proves that an ensemble of chaotic elements can not only effectively respond to a time-dependent input, but can even sustain it. An important step to understand how chaotic systems can respond to an external stimulus was made by considering the simple setup of uncoupled systems [2]. In such conditions, the nonzero average response is the result of a modulation of the probability density, induced by the input signal [3,4]. Understanding when and how the response is large and reliable is however a formidable task. A simple case is that of a negative conditional Lyapunov exponent, when a large response is the consequence of the mutual synchronization among the single trajectories [5]. By following this idea, Uchida, McAllister, and Roy[6] proposed to characterize the consistency of the response of a dynamical system in terms of the cross correlation between pairs of output time series. Here we propose a different approach, based on ensemble averages rather than on the behavior of single trajectories. In this way, we are able to detect weaker forms of synchronization.

The most important analytical result has been derived by Ruelle [7], who showed that in the small input limit, the linear response of a hyperbolic system can be decomposed into two distinct contributions arising from the motion along stable and unstable directions, respectively. Such predictions have been recently confirmed by the numerical investigation of a neural network [8].

However, apart from the attempt, within statistical mechanics, of establishing a connection between linear response theory [9] and the underlying chaotic dynamics [10] and from the many papers devoted to unravel various forms of synchronization, very few studies have been devoted to characterizing the response of generic chaotic systems, and in particular, we are not aware of any experimental investigation. Here we contribute to fill the gap, by analyzing experimentally the behavior of an injected semiconductor laser and, numerically, two prototypical chaotic systems, a Hindmarsh-Rose neuron and the Rössler oscillator. Besides exploring the connection between response and synchronization, we present a quantitative study of the stability of the response. At variance with the "microscopic" stability of single trajectories, which is quantified by the maximum Lyapunov exponent, the "macroscopic" stability is determined by the spectrum of the Liouville operator $\mathcal{L}$, which describes the evolution of the corresponding probability density in the phase space. The direct investigation of $\mathcal{L}$ is a numerically hard task, as it requires working in functional spaces. On the other hand, here we show that its first nonzero eigenvalue $-\gamma_1$ can be efficiently determined by following ensembles of distinct trajectories. Notice that this coordinate-independent observable provides the most important information on the stability, as it corresponds to the inverse of the longest time scale.

The semiconductor laser.—The experimental setup is a standard optical injection configuration. The slave laser is a commercial edge-emitter semiconductor laser, namely, a Roithner RLT 98005 MG single transverse, multilongitudinal mode, lasing at 980 nm. The slave is injected by a tunable beam from a (master) laser mounted in an external grating configuration (Littrow), with an emitting power of 150 mW. The beam is spatially filtered and amplified, and the optical power intensity at the slave laser is about 5 mW. Such intensity can be easily changed; the injected level in the regime we study is between 2 and 4 mW. In the chaotic regime the slave is locked to the frequency injected and exhibits a broad peak in the optical spectrum whose width is around one-third of the free spectral range of the laser (75 GHz). The slave is mounted in a commercial head, with a modulation cutoff at about 1 GHz. The input signal is generated by a fast (up to 3 GHz) Hewlett Packard pulse generator and summed to the slave pump current. The output slave intensity is detected by a fast (about 8 GHz) detector and acquired by a LeCroy digital scope (6 GHz bandwidth, 20 gigasamples/ sec). The input signal is split and acquired synchronously, for calibration purposes. The scope memory allows us to record up to 8×10^6 samples per measurement, both for input and output signals. The laser threshold is $I_{\text{th}} = 15$ mA, while the bias current is $I_B = 23$ mA. The amplitude modulation is $I_m = 4$ mA peak-to-peak, corresponding to a modulation depth of $100 \times I_m/(I_B - I_{\text{th}}) = 50\%$ on a working point of $100 \times (I_B - I_{\text{th}})/I_B \simeq 53\%$ over threshold.

We perform a set of measurements by sending a pulsed, periodic waveform (500 MHz modulation frequency) with different duty cycles (DCs). As a result, the frequencies emerging from the DC modulation, naturally fall in the chaotic laser band which extends from about 1 to 1.5 GHz. The output signal from the detector is sampled at 20 gigasamples/ sec , thus obtaining a pseudoensemble of 2×10^5 elements, each of 40 samples. The resulting waveforms are treated as statistically independent responses to replicas of the same input. In Fig. 1(a) we show one period of four different input waveforms and their statistical fluctuations, which are sufficiently small to conclude that the laser is coherently driven. The average response in the four cases is plotted in Fig. 1(b), where one can appreciate the qualitatively relevant differences. As a check of the reliability of the curves, in Fig. 1(c) we show the convergence of the average signal for increasing numbers of ensemble elements for DC = 80%. The amount of fluctuations of the averaged waveforms is indeed very small and allows us to consider them as a well-defined response of the system to the different input signals. We remark that from the time-resolved, ensemble distributions (two examples are shown in the inset in Fig. 1(b) for the 2 times at DC = 60%) other averaged quantities can be easily obtained to characterize the response, e.g., standard deviation, higher-order moments, etc.

In order to investigate the possible presence of phase synchronization (PS), we have Hilbert transformed [5] both the input and the output signals. The results for the four different DCs reveal that the frequency difference $\Delta\Omega$ between input and output varies linearly with the DC (see the inset of Fig. 2) and, except for DC = 80%, there is no frequency synchronization. By comparing $\Delta\Omega$ with the root-mean-square (rms) of the average response [see Fig. 1 (b)], it is clear that the amplitude of the response is not connected to the existence of frequency synchronization; it

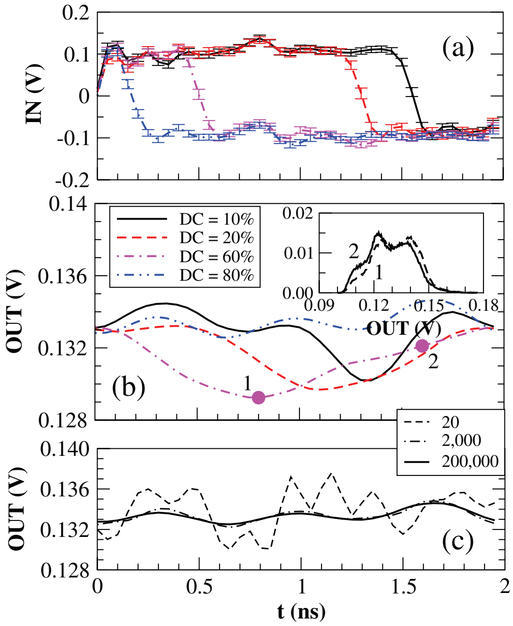

FIG. 1 (color online). Experiment. (a) Input waveforms, with different DCs. The error bars measure the fluctuations in the ensemble of input signals. (b) Output (averaged) waveforms for the corresponding inputs. In the inset are the ensemble distributions of intensity at times (1)–(2) for DC = 60%. (c) Convergence of average in the case of DC = 80%, for increasing the number of ensemble elements.

even turns out that the most synchronized signal is characterized by the smallest rms (less than half of the others). Once the average frequency is subtracted from the Hilbert transform, we can appreciate the fluctuations of the phase ϕ which turns out to diffuse for all four DCs we have studied. An example is shown in Fig. 2 for DC = 60%, where the phase of input and output are plotted together. This excludes also the existence of PS.

Hindmarsh-Rose model.—A more detailed and quantitative analysis can be carried out in mathematical models. We have first selected a periodically forced Hindmarsh-Rose (HR) model [11],

$$\dot{x} = x^2(3 - x) + y - z + a - \varepsilon x + KG(t),$$
$$\dot{y} = 1 - 5x^2 - y, \qquad \dot{z} = c(4x + 6.4 - z), \tag{1}$$

as this is a prototypical system for analyzing information 5
processing in neural networks. We have first considered an amplitude symmetric $(-1, +1)$ rectangular modulation $G(t)$ with variable DC. We have chosen to work in a parameter region where the HR neuron is known to behave chaotically. The modulation period has been selected close to that of spontaneous collective oscillations [12] to be sure of the responsiveness of the HR neuron. The results are plotted in Fig. 3, where the ensemble average $\langle x \rangle$ of the component x is plotted for DC = 10%, 20%, 80%, 90%.

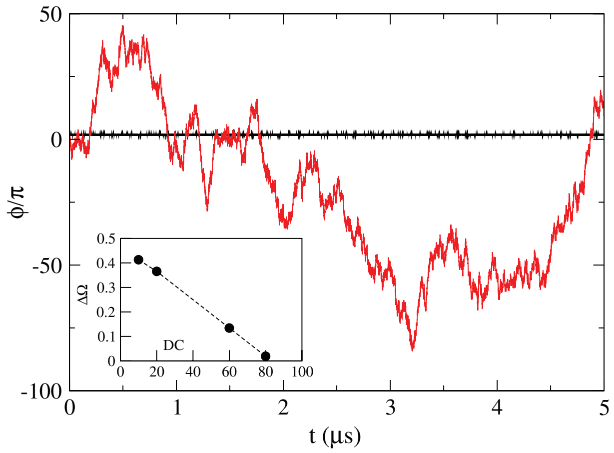

FIG. 2 (color online). Experiment. The phase of the input (flat curve) and output signal for DC = 60% as determined by means of the Hilbert transform, after subtracting the average frequency. In the inset the frequency difference between input and output is plotted versus the DC.

Analogously to the experimental setup, we see that modifications of the DC induce qualitative changes in the shape of the response. The high-frequency oscillations are real; they are the remnants of the spiky behavior of the single trajectories.

In order to test whether a large response is connected to some form of synchronization, we consider also sinusoidal modulations $G(t) = \cos(\omega t)$. In Fig. 4(a), the amplitude R of the response (defined as the ratio between the rms of the output and that of the input—$K/\sqrt{2}$) is plotted versus K for the same frequency as before. The strong deviations from a flat curve indicate that we are in a deeply nonlinear regime, including the peak close to $K = 0.001$, which corresponds to the onset of a perfectly synchronized behavior. With the exception of the peak itself, everywhere else there is not even PS, as we have checked by taking a proper Poincaré section. Yet the average response is rather large.

At variance with the experimental setup, here we can study other observables. In particular, it is possible to investigate the macroscopic stability of the average response. The starting point is the setup proposed in [13], where the authors qualitatively characterized the convergence of the invariant measure. Given a point $\mathbf{x}_1 = (x, y, z)$ and the phase of the forcing term, an ensemble of N different initial conditions is randomly generated in a box $\mathcal{B}$ of size Δ centered around $\mathbf{x}_1$. The response $\langle \mathbf{x}_1 \rangle$ at time t is thereby defined as the average of the three coordinates. By repeating the procedure for another ensemble of initial conditions distributed around the point $\mathbf{x}_2$, we obtain a second average trajectory $\langle \mathbf{x}_2 \rangle$. The Euclidean distance δ between $\langle \mathbf{x}_1 \rangle$ and $\langle \mathbf{x}_2 \rangle$ is expected to converge to zero with a rate equal to the largest nonzero eigenvalue of the corresponding evolution operator $\mathcal{L}$ without the need to directly evolve it. In the inset of Fig. 4(a) we show the behavior of δ, computed by averaging over $N = 4 \times 10^5$ for two different pairs of boxes. In order to guarantee a sufficiently large initial distance between the boxes, we have chosen

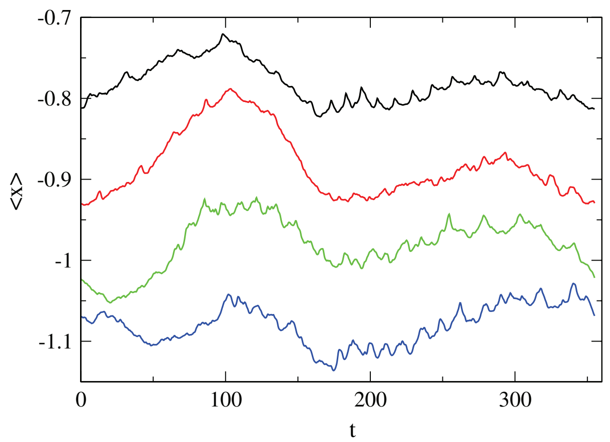

FIG. 3 (color online). Simulation. Average response of the Hindmarsh-Rose model (1) to periodic rectangular modulations with period $T = 2\pi/\omega$ ($\omega = 0.0177$) and different DCs: 10%, 20%, 80%, 90%—from top to bottom (curves have been vertically shifted for clarity). Averages are performed over 500 000 trajectories. The statistical error bar (not shown) is significantly smaller than the visible oscillations. The other parameters are $a = 3.3022$, $c = 0.0021$, $\varepsilon = 0.01$, $K = 0.001$.

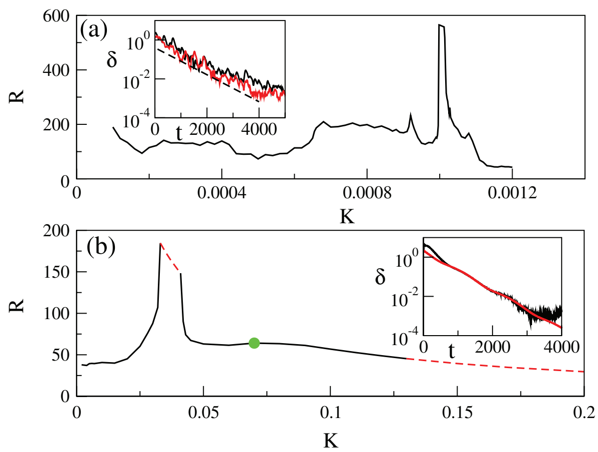

FIG. 4 (color online). Simulation. Response to a sinusoidal modulation. (a) A HR neuron with the same parameter values as in the previous figure. The dashed line identifies the synchronization regimes: everywhere else there is no synchronization. Inset: the distance between two different pairs of mean trajectories (average performed over 4×10^5 initial conditions) for $\omega = 0.015$, $K = 0.001$, and the other parameters as in the previous figure. The straight line corresponds to an exponential decay with a rate 0.0016. (b) Rössler oscillator with the parameter values defined in the text. Inset: the distance between two mean trajectories (average performed over 4×10^5 initial conditions) for the parameter values corresponding to the full dot in the main body of the figure.

their centers as the two maximally distant points on the attractor [14]. A clear exponential decay with a rate 0.0016 is visible before the saturation dictated by statistical fluctuations sets in. This is more than 1 order of magnitude smaller than the maximum Lyapunov exponent ($\lambda = 0.028$).

Rössler model.—In order to further clarify the relationship between a large response and (phase) synchronization, we have studied also a forced Rössler oscillator, the typical reference model in synchronization problems,

$$\dot{x} = -y - z + K\cos\omega t, \quad \dot{y} = x + 0.2y, \quad \dot{z} = 1 + z(x - 9). \tag{2}$$

In Fig. 4(b), we plot the rms of $\langle x \rangle$ normalized to the rms of the input, versus K. We observe large R values, as the modulation frequency ($\omega = 1.01$) is almost resonant with that of the unperturbed Rössler attractor. Analogously to the HR neuron, we are far from a linear regime. The most interesting observation concerns the connection with PS, whose presence is signaled by a dashed curve. It is intuitively reasonable to expect a large response in the presence of phase coherence. However, in Fig. 4 one can see that this is only qualitatively true, since [leaving aside the locking phenomenon associated to the peak in Fig. 4(b)] the highest response does not occur inside the PS region. In fact, one can have a large average response in spite of the presence of a fraction of trajectories that continuously lose synchrony. In the vicinity of the maximum response, for $K = 0.07$, we have studied the stability of ensemble averages by following the same protocol described for the HR neuron. In the inset of Fig. 4(b) we see that the distance δ between a pair of average trajectories (computed as for the HR neuron) decays exponentially with a rate $\gamma_1 = 4.6 \times 10^{-3}(1 + i)$ (where i denotes the imaginary unit)—that is again 1 order of magnitude smaller than the maximum (and only positive) Lyapunov exponent $\lambda_1 = 0.067$. We conjecture that, in both cases, the smallness of γ_1 is to be attributed to a slow relaxation of the phase dynamics and that, accordingly, a relatively small γ_1 is suggestive of the propensity for a dynamical system to exhibit PS. A more precise test of this idea would require going beyond the computation of the largest eigenvalue of $\mathcal{L}$, and determining, e.g., the first eigenvector of the corresponding Liouville operator. Computationally, this is a much harder task, which we plan to undertake in the future.

Altogether, in this Letter we have investigated the average response of various chaotic systems (including a laser experimental setup) in the presence of a relatively strong modulation. We have found that the response amplitude can be used as a tool to quantify broad forms of synchronization that go beyond, e.g., PS, without the need of identifying *a priori* a phase variable. We have also quantified the stability of the response, by determining the largest nonzero exponent $-\gamma_1$ of the Liouville operator. In both the Hindmarsh-Rose neuron and the Rössler oscillator, γ_1 turns out to be relatively small compared to the maximum Lyapunov exponent. This suggests the existence of a degree of freedom that is highly susceptible to external modulations and we conjecture that the smallness of γ_1 is at the root of the large response observed in such two systems. For the future, it will be important to be able to 6
determine the eigenvectors as well since this information will be useful in clarifying the issue. Finally, we have observed that qualitatively different responses are found for different duty cycles of the external modulation. This substantiates the hypothesis that the average response may be used for information encoding. However, it is necessary to shed some light on the functional relationship between the shape of the output and that of the input; it is from there that the potentiality of information processing in ensembles or networks of oscillators can be eventually demonstrated.

G. G. and A. P. wish to acknowledge financial support from the EU GABA Project No. FP6-NEST 043309; G. G. also acknowledges support from CNRS during his visit at INLN.

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