Fractal Dimension of Space-Time Chaos

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A class of simplified measures is constructed to capture the key features of generic spatiotemporally chaotic systems. A combined analytical and numerical investigation allows us to establish the scaling behavior of the fractal dimension in open systems. Our results improve a previous conjecture and, what is more important, furnish a clear framework for both numerical and analytical checks of the underlying assumptions.

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The structure of the invariant measure in generic spatiotemporal chaotic systems is still far from being understood. The only basic conclusion that is undoubtedly accepted within the physicist community is the extensivity of the fractal dimension, i.e., that the number of active degrees of freedom is proportional to the system size [1]. The number of degrees of freedom per unit volume is the so-called dimension density, a quantity that can be derived from the Lyapunov spectrum, through the well known Kaplan-Yorke formula.

However, as soon as finite subsets of (in principle) infinite systems are considered (i.e., an open-system point of view is adopted), it is immediately far from obvious how to characterize the probability distribution of generic observables. The common belief is that by looking at the system with a sufficiently coarse-grained resolution, one is not able to distinguish between a closed and an open system [1,2]. The stochastic-like action of the external world (i.e., the rest of the system) which activates otherwise stable degrees can be resolved only if we look at the dynamical system with a sufficiently high resolution. This was first suggested by Pomeau [3], who postulated that the effect of distant (in real space) degrees of freedom on the dynamics in a specific site decreases exponentially fast with the distance. However, even though this statement looks rather plausible, no convincing argument has yet been presented, which justifies this hypothesis in terms of the actual behavior of perturbations.

The same problem has been considered in the spirit of time-series analysis [1,4,5]. In particular, Tsimring [5] formulated a conjecture about the amount of information contained in finite samples of temporal signals generated by spatiotemporal chaotic regimes. In fact, this problem logically descends from the understanding of the structure of the invariant measure in open systems, since the behavior of a local observable over a finite time can be obtained by applying the evolution operator to typical spatial configurations in a finite region (the so-called light cone). For this reason, we shall restrict our discussion to this latter context in which Korzinov and Rabinovich [6] have proposed a conjecture similar to that in Ref. [5].

Specifically, they claim that the effective (i.e., finite scale) dimension $D_e$ in subsystems of length $L$ (for the sake of simplicity, we consider one-dimensional systems) depends on $L$ and on the observational scale $\varepsilon$ as

$$D_e = dL - \frac{\nu d^2}{\eta} \ln \varepsilon - A, \quad (1)$$

where $d$ is the dimension density, $\eta$ is the Kolmogorov-Sinai entropy density [1], $\nu$ is the propagation velocity of disturbances, while $A$ is a non-better-specified parameter. Unfortunately, the derivation of the above formula is based on several assumptions that cannot be directly checked. Moreover, it is rather unlikely that more accurate numerical simulations will ever provide data clean enough to draw definite conclusions. It is therefore compelling to make some progress on the theoretical side even at the expense of introducing strong simplifications. This is the route already undertaken in Ref. [7], where the limit case of weakly coupled maps has been considered. Here, rather than attempting to prove that the invariant measure of some dynamical system has a given structure, we have preferred to construct a class of measures fulfilling the basic requirements for a space-time chaotic regime and yet are simple enough to be handled and controlled. The key approximation consists in assuming that the support of the probability distribution is a linear subspace, so that we can use a global approach such as singular-value decomposition to get at once information on the structure of the invariant measure on all possible length scales. Singular-value decomposition as a tool for analyzing space-time chaos has been already profitably applied to experimental [8] as well as to numerical [9] data, allowing one to identify the relevant modes. However, the validity of these results is limited by the unavoidable presence of nonlinearities which definitely induce a bending of the support as well as local nonuniformities in phase space. Here, by referring to suitable linear subspaces, we automatically get rid of these effects and can use a global methodology to extract local information.

We shall consider a scalar process $x(i)$ defined on a 1D lattice of length $N$, with the label $i$ denoting the spatial...
position. A generic configuration is constructed as the linear combination of some modes
\[ x(i) = \sum_{k=1}^{D} p_k e_k(i) , \]  
where the \( p_k \)'s are independent, identically distributed random variables and \( e_k(i) \) denotes the \( i \)th component of the \( k \)th mode. In other words, the coefficients \( p_k \) represent the coordinates of the state \( (x(i)) \) in the basis of modes \( e_k \). The number of modes \( D = dN \) can be read as the “fractal” dimension of the measure, while \( d \) is the dimension density which is assumed to be independent of the length, i.e., an intrinsic characteristic of the underlying “dynamical system.”

For the sake of simplicity, we first restrict our analysis to Fourier modes,
\[ e_k(i) = \begin{cases} 1/\sqrt{N}, & \text{if } k = 1 , \\ \sqrt{2/N} \cos(\alpha_k i), & \text{if } k > 1 \text{ and odd} , \\ \sqrt{2/N} \sin(\alpha_k i), & \text{if } k \text{ even} , \end{cases} \]
where \( \alpha_k = 2\pi[k/2]/N \) and \([\cdot]\) denotes the integer part.

Even if very simple, such a hypothesis is not simplistic. In fact, let us recall that Fourier modes provide the stable or unstable directions in a chain of Bernoulli maps \( x_{i+1} = \text{mod}[a(\mu x_i(i-1) + (1-2\mu)x_i(i) + \mu x_i(i+1), 1) \] if one chooses the local slope \( a \) and the coupling strength \( \mu \) such that the Kaplan-Yorke dimension density \([1]\) is equal to \( d \), the active degrees of freedom correspond precisely to the above-mentioned \( D \) Fourier modes.

The problem we want to address concerns the structure of the projection \( S^N(L, d) \) of the global invariant measure onto the lower-dimensional space corresponding to a subchain of length \( L \ll N \). We expect that the resulting distribution extends along all \( L \) directions, i.e., that its dimension coincides with the space dimension \( L \). Nevertheless, we also expect \( S^N(L, d) \) to be very thin along those directions corresponding (in the closed system) to the most contracting directions (i.e., those modes \( \tilde{e}_k \) with \( k > D \) \([10] \).

As we are dealing with a linear subspace, the extension of \( S^N(L, d) \) along the various directions can be obtained by means of the standard orthogonal decomposition of the correlation matrix
\[ C_{ij}^N(L, d) = \langle x(i), x(j) \rangle = \sum_{k=1}^{D} \langle p_k^2 e_k(i) e_k(j) \rangle , \]
where \( \langle \cdot \rangle \) denotes an ensemble average and the last equality can be easily proved by substituting Eq. (2). For the sake of simplicity we introduce the otherwise arbitrary assumption that the \( p_k \)'s are uniformly distributed within the interval \([-3^{1/3}, 3^{1/3}]\), so that \( \langle p_k^2 \rangle = \int_0^{3^{1/3}} x^2 \, dx = 1 \).

It is possible to obtain an analytic expression for the correlation matrix. By substituting Eq. (3) into Eq. (4), and assuming that \( D \) is odd, we find, after some simple manipulations, that
\[ C_{ii}^N(L, d) = \frac{1}{N} + \frac{2}{N} \sum_{k=1}^{(D-1)/2} \cos[(i - j)\alpha_{2k}] . \]

It is immediately seen that \( C_{ii} = D/N = d \), while a general expression for the other entries can be obtained by using the relation \( \sum_{k=1}^{N} \cos(kx) = \cos[(n + 1)x/2] \sin(nx/2)/\sin(x/2) \) and other simple trigonometric formulas,
\[ C_{ij}^N(L, d) = \frac{\sin[(i - j)\pi d]}{\sin[(i - j)\pi/N]} . \]

In the limit \( N \to \infty \) (i.e., for an infinitely extended system) but fixed dimension density \( d \) and observational length \( L \), the correlation matrix converges to
\[ C_{ij}(L, d) = \frac{\sin[(i - j)d \pi]}{\pi(i - j)} . \]

Similar calculations for even \( D \) show that \( C_{ij} \) converges to the same limit, which thus holds in full generality.

Since the matrix entries depend only on \(|i - j|\), the values along all diagonals are constant (Toeplitz structure) and the correlation matrix is completely determined by its first row or column. Although this structure is rather simple, all attempts to diagonalize analytically the correlation matrix failed. Therefore, we have been obliged to perform numerical investigations with high precision \((10^{-32})\).

The most convenient way to look at the results is by ordering the eigenvalues \( \lambda^2_i \) from the largest to the smallest one. The qualitative behavior for \( d = 0.5 \) and \( L = 20 \) can be seen in Fig. 1, where we have reported

![FIG. 1. Extensions along the orthogonal directions for an object with dimension density \( d = 0.5 \) and an embedding dimension \( L = 20 \).](image-url)
the linear extensions $\lambda_i$ of the projected measure along the orthogonal axes represented by the eigenvectors of the correlation matrix. We see that approximately 10 values are close to 1, while the others are almost negligible. This confirms that, on a coarse-grained scale, an open system looks like a closed system with the same dimension density $d$. Nevertheless, the eigenvalues beyond the 10th are not equal to zero; this is the result of the coupling with the pseudorandom process generated by the external chain.

From the viewpoint of fractal dimension analysis, we can interpret these results by saying that the effective dimension corresponding to a fixed observational resolution $\varepsilon = \lambda_i$ is $D_\varepsilon = i$. The expression for the effective dimension can thus be obtained by inverting the expression for the eigenvalue distribution,

$$D_\varepsilon(\varepsilon) = i(\lambda).$$

(8)

In the specific case reported in Fig. 1, the dimension seen for $\varepsilon \approx 0.1$ is still smaller than 13. Only for $\varepsilon$ values as tiny as $10^{-7}$, the full space dimension (20) is recovered.

This problem has a meaningful interpretation also in the context of linear time-series analysis, since we are addressing the question of how the average Fourier spectrum of a stochastic signal (observed in a window of length $L$) converges to the limit shape for increasing $L$. The difference is that here the spectra of the signals are obtained by means of singular-value decomposition rather than finite Fourier transform.

Our main goal is to extract the relevant asymptotic behavior of $D_\varepsilon$ on the window length. As it can be seen in Fig. 2, where $\ln \lambda_i/L$ is plotted versus $i/L$ for different choices of $L$, there is a clear evidence of an asymptotic scaling regime such as

$$\ln \lambda_i = -LF(i/L),$$

(9)

where the function $F(x)$ is identically zero for $x < d$, while it increases monotonously for $x > d$. From a fundamental point of view, it is important to have a clear idea about the “critical” behavior for $x = d$. In particular, it is necessary to understand what kind of singularity is present in that region. The numerical analysis indicates that the right derivative $F'(x)$ computed in $x = d$ converges to a finite value. Accordingly, for $x > d$, we can expand $F(x)$ in a power series,

$$F(x) = \sum_{j=1}^{\infty} \beta_j (x - d)^j.$$

(10)

By substituting Eq. (10) in Eq. (9) and retaining only the first two terms we find the approximate expression

$$\ln \varepsilon = -\beta_1(D_\varepsilon - d) - \frac{\beta_2}{L} (D_\varepsilon - d)^2,$$

(11)

where we have identified $\lambda$ with $\varepsilon$ and $i$ with $D_\varepsilon$. Before commenting on the physical implications of the above relation, it is important to discuss finite-size corrections. A technical analysis exploiting the symmetry properties of $\lambda_i^2$ suggests that the deviation from the asymptotic value of $F(x)$ around $i = dL$ is equal to $-(\ln 2)/2L$ [12].

By including such corrections in Eq. (11) and thereby inverting, we find that

$$D_\varepsilon = dL - \ln \varepsilon - \frac{\beta_2}{\beta_1^3L} \ln^2 \varepsilon - \frac{2}{2}.$$

(12)

The structure of this expression reduces to that of the conjecture in Ref. [6] [Eq. (1)] provided that the squared logarithmic term is negligible, i.e., provided that $L \gg \ln \varepsilon$. One can easily check that this inequality is not always satisfied in direct computations of the dimension, so that it would be worth reanalyzing numerical data in order to obtain reliable estimates of the various coefficients.

A further difference with Eq. (1) resides in the coefficient in front of the logarithmic term: we find $1/\beta_1$ that is to be confronted with $vd^2/\eta$. We are able to determine $\beta_1$ only numerically (even though rather accurately) and it turns out to be approximately equal to 0.6 independently of $d$ in the interval [0.2, 0.8]. The simplest way to compare Eq. (1) with our expectations is again by referring to the chain of Bernoulli maps in which case one can determine analytically the Lyapunov spectrum [11] and easily compute $d$, $\eta$, and $\nu$. Tests made for different choices of the slope $a$ and of the diffusive coupling $\mu$ show that $vd^2/\eta$ is always significantly larger than $1/\beta_1$ (by almost a factor of 2). This inequality indicates at least that no gross inconsistencies are present. In fact, in the worst case Eq. (12) can be viewed as a lower bound for the effective dimension, since it has been obtained by neglecting nonlinearities (which enter as discontinuities in Bernoulli maps) which can only contribute to increase the dimension on finite scales. Anyhow, it would be very
important to establish whether the discrepancy is to be attributed to a failure of the conjecture in Ref. [6], or to an ingredient neglected in our treatment. In favor of the former hypothesis, we claim that, as long as nonlinearities imply only large scale phenomena such as bendings and nonuniformities, they should not affect the asymptotic behavior described by the scaling function $F(x)$. Indeed, from the point of view of an information-theoretic approach (which is implicitly that one adopted by deriving an expression for the effective dimension), there is no difference between a straight and a bended manifold, provided that the bending is not too strong [13]: the same number of boxes is needed to cover the set.

Aside from the role of nonlinearities, the validity of our results could be challenged on the basis of the special assumptions made to construct the invariant measure. Therefore, we have decided to progressively remove some of the limitations. We have started by assuming that the average amplitude of the first (nonzero) $D$ Fourier modes is not constant but goes continuously to 0. In this case, one can again find an explicit expression for the correlation matrix; numerical studies performed for different choices of the Fourier amplitudes indicate a convergence towards the same function $F(x)$ but stronger finite-size corrections (i.e., larger $A$).

A further criticism could be that the Fourier basis is a rather special choice, contrasting with the observation that Lyapunov vectors are localized [14]. We have therefore considered also bases made of exponentially localized vectors. This has been done by implicitly referring to Bernoulli maps with quenched disorder (i.e., by randomly fixing the slope of the local maps). In this case, the correlation matrix can only be constructed numerically and for lattices of finite length $N$; moreover, there is the additional difficulty of the dependence on the realization of the disorder. Surprisingly enough, we find again a reasonable agreement with the results obtained for the Fourier basis with the same system size (provided that the geometric average of the eigenvalue spectra is taken). Accordingly, we are led to conjecture that the behavior displayed by the Fourier basis is quite universal and seemingly independent even of the localization properties of the basis.

In summary, we have found a rather general scaling law expressing the dependence of the effective dimension on the size of an open system and on the observational resolution. As the result has been derived under the assumption of a linear structure for the invariant measure, it is now crucial to test its validity in generic models, where nonlinearities certainly play an important role. Finally, we mention some recent rigorous results about the leading correction to the effective dimension in a 1D complex Ginzburg-Landau equation [15]. The upper bound derived in Ref. [15] is consistent with the results obtained in the present Letter.

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[10] Let us indeed recall that in Bernoulli maps, Fourier modes with increasing wave number are increasingly stable if $\mu < 0.5$ (see Ref. [11]).
[13] The problem of accounting for the possibly Cantor-like structure along some directions would be different. However, if this phenomenon is restricted to just one direction, as implicit in the Kaplan-Yorke formula, it should not imply relevant corrections to our analysis.