High-dimensional chaos in delayed dynamical systems

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Received 9 February 1993
Revised manuscript received 23 June 1993
Accepted 9 August 1993
Communicated by K. Kaneko

We introduce a general class of iterative delay maps to model high-dimensional chaos in dynamical systems with delayed feedback. The class includes as particular cases systems with a linear local dynamics. We report analytic and numerical results on the scaling laws of Lyapunov spectra with a number of degrees of freedom. Invariant measure is computed through a self-consistent Frobenius–Perron formalism, which allows also a recalculation of the maximum Lyapunov exponent in good agreement with the one measured directly.

1. Introduction

In many physical situations the causality principle imposes the inclusion of retarded actions. The dynamics is thus modeled by differential-delay equations of the form

\[ \dot{y}(t) = F(y(t), y(t-\tau), \mu), \]

where \( F \) is a nonlinear function, \( \tau \) is the delay time, \( \mu \) is a control parameter and the state variable \( y \) is \( N \)-dimensional. Many different models have been introduced in various domains such as optics, biology and physiology. In almost all of the systems, nonlinear interactions pertain only to the delayed feedback of a scalar variable \( y \),

\[ \dot{y}(t) = -\gamma y(t) + f(y(t-\tau), \mu). \]

This type of models, referred to in the following as class-I systems, includes the Mackey–Glass equation [1]

\[ \dot{y}(t) = -y(t) + \frac{\mu y(t-\tau)}{1 + y^{10}(t-\tau)}, \]

introduced to describe the creation of blood cells, and the Ikeda equation [2]

\[ \dot{y}(t) = -y(t) + \pi \mu \sin[y(t-\tau) - x_0], \]

which is a model of a nonlinear optical resonator. Notice that in both cases the parameter \( \gamma \) has been scaled out.

Farmer [3], with reference to model (1.3), has numerically studied the asymptotic dependence on \( \tau \) of various chaotic indicators. He showed that the Lyapunov spectra decrease as \( 1/\tau \), while the metric entropy converges to a finite value and the fractal dimension is an extensive quantity proportional to \( \tau \). His findings have been later confirmed by Ikeda and Matsumoto [4] who investigated eq. (1.4). They have also analyzed the hierarchy of bifurcations which, upon increasing the control parameter \( \mu \), lead to a high-dimensional chaotic attractor. Finally, they have
studied the structure of the attractor both by means of a Lyapunov-vector analysis, and by applying information-theoretic concepts to Fourier space [5].

At variance with eq. (1.2), we expect, in general, a nonlinear instantaneous (local) coupling to be also present. A remarkable example has been recently studied in [6]: it consists of a CO$_2$ laser with delayed feedback on the cavity losses. In the present paper we aim at exploring the dynamical properties of this more general class of dynamical systems. However, in order to facilitate the numerical investigation, we wish to restrict ourselves to the discrete time case. Our choice is indeed motivated by the same spirit which inspired the introduction of coupled map lattices [7], where the discretization of both space and time variables, while preserving many of the characteristic features of the spatio-temporal complexity displayed by partial differential equations, made possible extensive simulations of such phenomena.

The shortest route leading to the introduction of discrete-time mappings is provided by the application to eq. (1.2) of the Euler integration scheme,

$$y_{i+1} = y(1 - \varepsilon)y_i + \varepsilon f(y_{i+1-T}, \mu).$$

(1.5)

Here, the time values (separated by $\delta$) are labelled by the integer variable $t$, and the control parameter $\varepsilon = \delta/\gamma$ replaces $\gamma$. The delay time is now controlled by $T = \tau/\delta + 1$.

A local nonlinear coupling can be simply introduced in eq. (1.5) by writing

$$y_{i+1} = (1 - \varepsilon)f_1(y_i) + \varepsilon f_2(y_{i+1-T}),$$

(1.6)

where $f_1$ and $f_2$ are two nonlinear maps of the interval, and $\varepsilon$ (0 $\leq$ $\varepsilon$ $\leq$ 1) controls the relative weight of the delayed coupling. For the sake of brevity, the dependence on further control parameters has been dropped. This type of systems, referred to as class-II models, will be the object of an extensive study in this paper. Before entering the discussion of the dynamical properties, we stress that eq. (1.6) should not, at this stage, be considered as the approximation of some differential delay equation. We rather wish to consider eq. (1.6) as a prototype of a more general class of delayed systems. The problem of matching discrete mappings with differential delay equations is a very interesting question which goes beyond the scope of the present analysis.

An important difference with continuous-time models is represented by the intrinsic finite dimensionality of the phase-space which, in our case, is just $T$. This simplification should be seen as an advantage rather than a limitation of the model, since also in differential-delay equations (characterized by an infinite-dimensional phase-space) the attractors turn out to be finite-dimensional, under rather general assumptions [8]. In fact, the nonzero correlation time $t_c$ induced by the dynamics suggests that only a finite number $n = \tau/t_c$ of independent samples are needed to define a point on the attractor. This argument has been further pursued in [9], where the link between the number of effective degrees of freedom and the decay rate of the correlation function has been investigated.

The low-dimensional attractors arising for small $T$ can be studied by means of standard techniques [10]. Conversely, in the long-delay limit, a more complex high-dimensional dynamics appears. It is our aim to investigate this latter regime. In particular, we will focus our study on the asymptotic dependence of the chaotic indicators on the delay time $T$, a problem which essentially corresponds to studying the thermodynamic limit of spatially extended systems.

Our analysis, while confirming some of the conjectures raised for class-I systems, reveals novel features too. The main questions arising in class-II systems are: what is the result of the competition between the nonlinearities due to $f_1$ and those due to $f_2$; is the scaling with the delay affected by $f_1$; to what extent is a delayed system equivalent to a spatially extended dynamical system; is there any limit situation whereby the
delayed contribution acts as a noise term, practically independent of the amount of the delay? An approach in this sense has been already developed in ref. [11] for class-I systems.

In section 2 we evaluate the Lyapunov exponents and show that, besides an ordinary spectral component scaling as \(1/\tau\), there is a point-like component which remains finite for \(T \to \infty\). In section 3 we discuss the correlation functions of the dynamical variable, showing evidence of a revival phenomenon characterized by large correlation peaks spaced by one delay unit. This phenomenon was also observed experimentally in the output intensity of the above mentioned laser system [12]. Correlation functions, as well as other statistical quantities, can be evaluated as ensemble averages, provided the invariant measure is known. The associated problem is dealt with in section 4. Application of the Frobenius–Perron formalism leads to an infinite sequence of equations entailing an open hierarchy of joint probability densities of increasing order. Its solution requires the introduction of suitable truncations. Two procedures are shown, namely an elementary one, whereby the joint density is factored out as the product of the single densities at different times, and a Markov one, where higher-order densities are expressed in terms of second-order ones.

Section 5 is a display of numerical results. In particular, the maximum Lyapunov exponent is recalculated by a mean-field approach based on the knowledge of the invariant measure. In the appendix the scaling behaviour is discussed of the invariant measure at the two extrema of the unit interval, where analytic results can be worked out.

2. Lyapunov analysis

A direct inspection of eq. (1.6) suggests the existence of two different regimes, (i) one for small \(\epsilon\), where the delay term represents a perturbation to the single map behaviour, and (ii) one for \(\epsilon\) close to unity, where the local nonlinearity is strongly damped by \((1 - \epsilon)\) and hence we are practically in a class-I regime. We will see that this qualitative demarcation is essentially confirmed by a quantitative investigation.

We start studying of the Lyapunov spectrum of model (1.6). The temporal evolution of an infinitesimal perturbation \(\delta y\), is described by the linearized map

\[
\delta y_{t+1} = (1 - \epsilon)f'_1(y_t) \delta y_t + \epsilon f'_2(y_{t+1-\tau}) \delta y_{t+1-\tau},
\]

(2.1)

where the prime denotes the derivative with respect to the argument. If, in the limit \(T \to \infty\), the exponential growth rate of \(\delta y\) remains finite, then the second term in the r.h.s. of eq. (2.1) can be neglected, and we are left with a purely multiplicative process. In such a case, the maximum Lyapunov exponent is simply given by

\[
\lambda_{\text{max}} = \langle \log |f'_1(y_t)| \rangle + \log(1 - \epsilon),
\]

(2.2)

where \(\langle \cdots \rangle\) denotes the temporal average. From now on we will define “anomalous” this exponent to stress its different scaling behaviour with respect to the rest of the spectrum and to all other cases so far discussed in the literature, where a \(1/T\) scaling is reported [3,4]. The delay term \(f'_2\) affects \(\lambda_{\text{max}}\) implicitly, contributing only to determine the trajectory along which the average is computed. Eq. (2.2) holds whenever the resulting Lyapunov exponent turns out to be positive, consistently with the initial assumption. It is clear that the existence of a strictly positive Lyapunov exponent even in the limit \(T \to \infty\) follows necessarily from the nonlinear character of the local coupling. This point can be discussed in the general framework of a set of \(N\) differential-delay equations such as (1.1). Positive Lyapunov exponents, asymptotically independent of \(\tau\), can be rigorously computed by regarding eq. (1.1) as a nonautonomous differential equation with the delay terms acting as “stochastic” contributions [9]. Indeed, for \(\tau\) large com-
pared with the inverse of each exponent, delay terms can be neglected in the linearized equations. Therefore the number of such exponents can at most equal the number N of equations, and it is in any case subject to the restrictions arising from the overall stability requirements of the local-time interactions. In the case of class-I equations, the linear and stable structure of $f_{\pi}$ prevents the existence of such exponents.

Let us now specialize our investigation to a case susceptible of analytic treatment, that is replace $f_{1}$ and $f_{2}$ with two Bernoulli shifts,

$$y_{t+1} = \lambda(1 - \epsilon) + \epsilon y_{t+1 - T} + \epsilon y_{t+1 - T},$$  \hspace{1cm} (2.3)

where $\lambda(y) = y \mod(1)$. Here, $a$ and $b$ are the slopes of the piecewise linear maps ($|a| > 1, |b| > 1$). Eq. (2.2) straightforwardly yields

$$\lambda_{\max} = \log|a(1 - \epsilon)|.$$  \hspace{1cm} (2.4)

The result is consistently positive for $\epsilon < \epsilon_{\zeta} = 1 - 1/|a|$. For larger $\epsilon$, the local coupling has a stabilizing effect and the maximum Lyapunov exponent cannot be computed via this method.

The same phenomenology is also found in the less trivial case of continuous maps. However, before displaying the results of numerical simulations, let us further simplify eq. (1.6), by assuming $f = f_{1} = f_{2},$

$$y_{t+1} = (1 - \epsilon) f(y_{t}) + \epsilon f(y_{t+1 - T}).$$  \hspace{1cm} (2.5)

Notice that, analogously to diffusively coupled maps, eq. (2.5) can be rewritten as

$$x_{t+1} = f((1 - \epsilon)x_{t} + \epsilon x_{t+1 - T}).$$  \hspace{1cm} (2.6)

The latter expression is obtained upon introducing $x_{t} = f(y_{t})$ and after applying the map $f$ to both sides of eq. (2.5).

The maximum Lyapunov exponent for the logistic map $f(x) = 4x(1 - x)$ is reported in fig. 1 versus $\epsilon$ (dashed line). The fall of $\lambda_{\max}$ around $\epsilon = 0.17$ is due to the presence of a stable window of period 2. It is worth recalling that the anomalous scaling of the maximum exponent stems from the local nonlinearity rather than from the delayed action. This effect is present only in the weak-coupling regime.

Independently of the scaling behaviour exhibited by the maximum exponent, the remaining part of the spectrum scales as $1/T$, for every value of the coupling constant. In the simple case of two Bernoulli maps (eq. (2.3)), it is possible to work out analytically the investigation of the whole spectrum. The linearized equation

$$\delta x_{t+1} = a(1 - \epsilon) \delta x_{t} + b \epsilon \delta x_{t+1 - T},$$  \hspace{1cm} (2.7)

is independent of time, owing to the fact that the derivatives are independent of $x$. We can then look for Laplace-type solutions of (2.7), $\delta x_{t} = e^{st}$, obtaining the eigenvalue equation

$$e^{s} = a(1 - \epsilon) + b \epsilon e^{-s(T-1)}.$$  \hspace{1cm} (2.8)

The Lyapunov exponents are nothing but the real parts $\lambda$ of the $T$ solutions $s = \lambda + i\omega$ of eq. (2.8). Each exponent is parametrized as a suitable monotonic function of the corresponding imaginary part $\omega$. Its explicit expression is obtained by separating real and imaginary parts of eq. (2.8) and assuming that, for very long delays, $\lambda \rightarrow \Lambda/T$, with $\Lambda = O(1)$. In such a case, since $e^{(T-1)/T} \approx e^{\Lambda}$, we obtain


\[
\cos[\omega(T-1)] = \frac{1}{b\epsilon} e^{A[\cos \omega - a(1-\epsilon)]}, \\
\sin[\omega(T-1)] = -\frac{1}{b\epsilon} e^{A \sin \omega}. 
\] 

(2.9)

By squaring and summing, we can solve with respect to \( A \), obtaining

\[
A(\xi) = \frac{1}{2} \log\left(\frac{b^2 \epsilon^2}{1 + a^2(1-\epsilon)^2 - 2|a|(1-\epsilon) \cos \pi \xi}\right), 
\]

(2.10)

where the index \( \xi = \omega/\pi \) (0 \( \leq \xi \leq 1 \)) is the integrated density of Lyapunov exponents. This is the consequence of the fact that \( \omega \) represents an asymptotically uniform parametrization of the solutions of eq. (2.9).

The extrema of the monotonic function \( A(\xi) \) are

\[
A_{\text{max}} = A(0) = \log \left| \frac{b \epsilon}{1 - |a|(1-\epsilon)} \right|, 
\]

(2.11a)

\[
A_{\text{min}} = A(1) = \log \left| \frac{b \epsilon}{1 + |a|(1-\epsilon)} \right|. 
\]

(2.11b)

It is easy to show that \( A_{\text{max}} < 0 \) for \( \epsilon < \epsilon_1 = (|a| - 1)/(|a| + |b|) \), while \( A_{\text{min}} > 0 \) for \( \epsilon > \epsilon_2 = (|a| + 1)/(|a| + |b|) \). The spectrum (2.10) then reveals three different regimes:

(i) \( 0 < \epsilon < \epsilon_1 \): the whole spectrum is negative;

(ii) \( \epsilon_1 < \epsilon < \epsilon_2 \): both positive and negative components are present;

(iii) \( \epsilon_2 < \epsilon < 1 \): the spectrum is completely positive.

The classification of distinct regimes can be further refined by recalling that, depending on whether \( \epsilon < \epsilon_c \) (\( \epsilon > \epsilon_c \)), an anomalous component is (is not) present. Accordingly, four different regimes can be always identified (notice that \( \epsilon_c \) can be neither smaller or greater than \( \epsilon_2 \)). In fig. 2, we show the various spectra obtained for \( a = b = 2 \).

In the small \( \epsilon \)-limit there is a 1D source of instability (the local coupling) which acts as a forcing term for the remaining \( T - 1 \) stable directions. In the opposite regime, we are in the limit of \( T \) almost independent dynamics, a situation resembling weakly coupled maps, if the time unit is rescaled to \( T \).

It is clear that the previous classification applies to any map in the vicinity of its fixed points: in such a case \( a \) represents the local slope (for the sake of simplicity let us again assume \( f_1 = f_2 \)). For instance, in the logistic map the stability of the points \( x = 0, x = \frac{3}{4} \) is accounted for by assuming \( a = 4, -2 \), respectively. The critical \( \epsilon \)-values separating the various regimes are obviously different for the two fixed points. The dependence of the critical point on the orbit is very likely to persist also for longer periodic cycles. Accordingly, it is not a priori obvious which would be the behaviour displayed by a generic chaotic trajectory. Numerical experiments suggest that the same classification scheme applies to the average Lyapunov spectrum. The results corresponding to two distinct \( \epsilon \)-values for the logistic map are reported in fig. 3.

Let us in particular investigate the transition where the anomalous component disappears.
From the analytic expression (2.11) of the Lyapunov spectrum, it is easily seen that the maximum exponent diverges logarithmically when approaching \( \varepsilon_c \) from above (\( \Lambda_{\text{max}} = -\log(\varepsilon - \varepsilon_c) \)). In order to test the generality of this scaling law, we have numerically iterated the delayed logistic map. The critical value \( \varepsilon_c \) can be accurately computed from eq. (2.4), by determining the \( \varepsilon \)-value where the anomalous exponent vanishes, a measure which is not affected by finite size corrections and gives \( \varepsilon_c = 0.3483 \ldots \).

The results of direct simulations, made above \( \varepsilon_c \), with a sufficiently long delay are reported in fig. 4. Since the best fit of the log–log plot has a slope very close to \(-1\), we can say that the divergence of \( \Lambda_{\text{max}} \) is of the type \( 1/(\varepsilon - \varepsilon_c) \).

From the knowledge of the Lyapunov spectrum, one can infer both the Kolmogorov–Sinai entropy \( K \) and the dimension \( D \) of the attractor. From the Pesin relation [13], it is known that the sum of all positive Lyapunov exponents provides an upper bound to the entropy (the bound is typically saturated). Therefore, in the long delay limit,

\[
K = \lim_{T \to \infty} \frac{1}{T} \sum_{i=1}^{i^*(T)} \lambda_i = \lambda_{\text{max}} + \lim_{T \to \infty} \frac{1}{T} \sum_{i=2}^{i^*(T)} \frac{\Lambda(i/T)}{T} \\
= \lambda_{\text{max}} + \int_{\xi^*}^{\xi} \Lambda(\xi) \, d\xi, \tag{2.12}
\]

where \( i^*(T) \) is the largest integer \( i \) such that \( \lambda_i > 0 \) and \( \lambda_{\text{max}} \) (i.e., \( \lambda_1 \)) represents the anomalous component. Finally, the existence of an asymptotic spectrum ensures that \( \xi^* = \lim_{T \to \infty} i^*(T)/T \) is finite. Therefore, at variance with coupled-map lattices where the Kolmogorov–Sinai entropy is an extensive quantity, here \( K \) remains finite in the thermodynamic limit \( T \to \infty \). Let us finally remark that in eq. (2.12) we have, as usual, dropped the dependence of the Lyapunov exponent \( \lambda_i \) on the initial condition, since \( \lambda_i \) takes almost everywhere in phase-space the same value (provided that no multiple attractors simultaneously exist).
In fig. 5 the dependence of $K$ on $\varepsilon$ is plotted for the Bernoulli map and two different choices of the parameters $a$, $b$. Although the integrand in eq. (2.12) is analytically known from eq. (2.10), a closed form for the integral is not available. The behaviour exhibited by the Kolmogorov–Sinai entropy can be qualitatively understood by recalling that in the limit $\varepsilon \to 0$ we expect $K$ to converge to the entropy associated with the map describing the local-time coupling, while for $\varepsilon \to 1$, the convergence to the entropy of the delayed coupling is expected. However, in the cases $a = b = 2$ not only the two opposite limits coincide, but an overall symmetry around $\varepsilon = 0.5$ is found (solid line). The same symmetry is again found in the simulations with the logistic map, when two period-2 stability windows appear as well (see fig. 6). The symmetry is a consequence of the choice of model (2.5), which treats on the same footing local and delayed couplings. However, it is by no means a trivial task to prove it rigorously even in the case of Bernoulli maps, since the symmetry is restricted to some but not all chaotic indicators. Recall, for instance, that the anomalous Lyapunov component does not exist for $\varepsilon > \varepsilon_c$. The density $d = D/T$ of the fractal dimension is also nonsymmetric. It can be implicitly computed through the Kaplan–Yorke formula [14]

$$\lambda_{\max} + \int_0^d \Lambda(\xi) \, d\xi = 0.$$  

(2.13)

The behaviour of $d$ is plotted in fig. 7 both for Bernoulli and logistic maps, where it is seen that the dimension increases monotonically with $\varepsilon$ (apart from the stability window). It saturates to 1 at large $\varepsilon$-values, where the system is no longer globally dissipative. This is a consequence of the noninvertibility of the dynamics which, in turn, is not merely due to our choice of the map $f_i$ but it is a general feature of models with delayed coupling. In differential-delay systems, the saturation of the dimension is not observed since the

![Fig. 5. Metric entropy $K$ for the Bernoulli map: $a = b = 2$ (solid line) and $a = 3.0$, $b = 1.5$ (dashed line).](image5)

![Fig. 6. Metric entropy $K$ for the logistic map.](image6)

![Fig. 7. Dimension density $d$: Bernoulli map with $a = b = 2$ (solid line), and logistic map (●).](image7)
integrated density of exponents $\xi$ is not bounded below 1 but, due to the infinite dimensionality of phase-space, diverges to infinity and, correspondingly, the Lyapunov spectrum becomes increasingly negative. More precisely, the linear stability analysis of class-I equations [3] shows that $\Lambda(\xi) \sim -\log \xi$ for $\xi \to +\infty$.

The opposite limit $\varepsilon \to 0$ is less trivial. By inspecting the analytical expression (2.10), it is seen that the whole Lyapunov spectrum diverges to $-\infty$. Therefore, it overcompensates the finite positive contribution coming from the single unstable direction associated with the anomalous component. Accordingly, the dimension $D$ tends to 1 and the density $d$ to 0.

According to the previous analysis, we see that in the infinite-delay limit the metric entropy remains finite, while the fractal dimension grows indefinitely. This seemingly paradoxical result, in agreement with previous conjectures raised for class-I systems [3,4], can be understood by realizing that $D$ does not depend on the absolute size of the Lyapunov exponents, but rather on the relative size of the positive with respect to the negative ones. A still better comprehension is obtained through a formal comparison with extended systems. The application of the $T$th iterate of eq. (2.5) maps a sequence of $T$ consecutive values $(x_t, \text{ for } 1 \leq t \leq T)$ onto the adjacent sequence $(x_{t+1}, \text{ for } T+1 \leq t \leq 2T)$. This simple observation permits immediately to recognize the analogy with chains of coupled maps, where one iteration maps an initial configuration (the state variables defined over $L$ lattice sites) onto the next one. More precisely, the time lattice can be decomposed into the sum of infinite delay units, so that a generic time $t$ can be represented by two integers $n (n > 1), i (1 \leq i \leq T)$, identifying the delay unit ("time" variable) and the position inside the delay unit ("spatial" variable), respectively. This is precisely the method used in ref. [12] to represent the experimental results. Therefore, after rescaling the time unit by $T$ in eq. (2.5), one would conjecture that the same scaling laws of extended systems should be recovered upon identifying the delay $T$ with the chain length $L$. Indeed, after rescaling the Lyapunov exponents by the factor $T$, the fractal dimension remains unchanged, while the metric entropy becomes an extensive quantity exactly as in coupled maps. The only exception to this analogy is represented by the anomalous exponents which, upon rescaling, appear as diverging quantities. The analogy between delayed and spatially extended dynamical systems used here to introduce the concept of thermodynamic limit in the former model, was indeed put forward several years ago in the context of hereditary systems [15].

3. Correlation functions

A more standard technique used in the investigation of irregular behaviours is based on the computation of the correlation function. This must be seen as a complementary approach, since the information contained in a correlation function does not provide a complete characterization of the dynamical behaviour. For instance, it is not possible to distinguish a stochastic process from purely deterministic chaos. In this sense, other methods, based on the computation of either mutual informations or fractal dimensions, are definitely more powerful. However, correlation functions are very useful in that they allow a direct and accurate comparison with experimental results also in the case of high-dimensional chaos, when other methods are inapplicable.

In the previous section, we have found two types of Lyapunov exponents, namely those ones remaining finite in the infinite-delay limit and those ones vanishing as $1/T$. It is interesting to see whether the decay rates of the correlation function exhibit both scaling behaviours.

Let us introduce the normalized autocorrelation function

$$C_t = \frac{\langle x_t x_{t+1} \rangle - \langle x \rangle^2}{\langle x^2 \rangle - \langle x \rangle^2}, \quad (3.1)$$
where \( \langle \cdots \rangle \) denotes the average over the running time variable \( s \). A rigorous analysis seems to be actually out of reach even in the case of a piecewise linear map, since there is no simple way to construct a Markov partition and then solve exactly the associated Frobenius–Perron equation. In the next section we present an approach which makes it possible to derive accurate, although approximate, expressions for the invariant measure. Here, we will be content with a numerical analysis.

Several simulations made for different delays show that, in general \( C_t \) is appreciably different from zero only around the multiples of the delay \( T \), when revival peaks are observed characterized by a width which remains finite for \( T \to \infty \). Some correlation functions obtained with the logistic map are summarized in fig. 8 for different \( \varepsilon \)-values. The only exception to this scenario is the plot of fig. 8 corresponding to a parameter-value \( \varepsilon = 0.16 \), i.e. within the stability window. It shows substantial nonzero \( C_t \)-values even away from multiples of \( T \). Notice that for \( \varepsilon = 0.80 \), successive peaks are anti-correlated, that is, they alternate in sign.

The presence of the correlation peaks can be clearly understood in the limit \( \varepsilon \to 1 \). In fact, in that limit, there are \( T \) approximately independent dynamics evolving along distinct time-lattices. In terms of the spatio–temporal variables introduced in the previous section, this corresponds to the situation where evolution on the “site” \( i \) is uncorrelated with that on \( j \) for \( i \neq j \). Accordingly, \( C_t \) is practically negligible for all \( t \)'s which are not multiples of the delay \( T \), while \( C_{mT} \) is close to the correlation of the single map after a time \( m \).

We are thus in presence of two distinct mechanisms:

(i) A slow decay occurring on the time scale of several delay units, which is responsible for the exponentially decreasing height of the peaks (see fig. 8).

(ii) A fast relaxation responsible for the finite width of each peak.

By pushing forward the analogy with extended systems, these two mechanisms can be identified with those responsible for temporal and spatial disorder, respectively.

The information on the amplitude of revival peaks for different values of \( \varepsilon \) can be presented in a simple and compact way, by plotting \( M_{C} = C_{tm} \), where \( t_m \) is the time corresponding to the maximum absolute value of correlation function, after the initial peak died out. As a consequence, \( M_C \) provides simultaneously amplitude and phase information about correlations. The results both for Bernoulli and logistic maps are reported in fig. 9, where it is seen that the height of the

![Fig. 8. Autocorrelation function for the delayed logistic map.](image)

![Fig. 9. Maximum peak height \( M_C \) of the autocorrelation function: Bernoulli map (dashed line) and logistic map (solid line).](image)
peaks is typically larger for the logistic map, except for ε close to 1. Notice that inside the period-2 windows of the logistic map, the strong correlation occurs with opposite sign, and this would not be inferred from the K-plot of fig. 6.

Finally, the study of correlation functions allows us to check the range of validity of a conjecture raised in ref. [7] on the fractal dimension D in class-I systems. There, it has been suggested that $D = T/t_c$, where $t_c$ is the correlation time. A careful analysis of the decay of the initial peak of $C_t$ in the logistic map shows that $t_c(ε)$ is roughly an increasing function of $ε$. This observation, combined with the results of the previous section, where we have seen that the dimension $D$ increases with $ε$ as well, is completely in contradiction with that conjecture. We suppose this failure to be strictly related to the chaotic nature of the local coupling. In fact, this source of irregular behaviour affects the short-time decay.

4. Frobenius–Perron formalism

A relevant point in the investigation of the dynamical behaviour of a given system is the estimation of the invariant measure which, in principle, makes the determination of all chaotic indicators possible. In high-dimensional dynamics, an exact estimation is typically out of reach. Therefore, it is important to develop approximation methods to estimate the probability density. Various approaches have been introduced to describe class-I differential-delay equations, either by following singular perturbation theory [16], or, more recently, by deriving a functional equation for the probability density [17]. In this section, we discuss the discrete-time mapping (1.6), showing that it is possible to write down an infinite hierarchy of equations. A mean-field solution is then obtained after introducing a self-consistent approximation, which allows us to truncate the hierarchy. The lowest-order approximation corresponds to assuming that (in the long $T$ limit) the delay term in the r.h.s. of eq. (2.5) is essentially uncorrelated with the local term. As a result, we will be able to determine the stationary single-time probability density and, eventually, to estimate the anomalous Lyapunov exponent. Our method is in a sense analogous to a procedure developed for coupled maps [18].

From eq. (2.6), it is seen that the probability density $ρ^{(1)}(x, t + 1)$ of variable $x$ at time $t + 1$ can be determined from the joint probability density $ρ^{(2)}(u, t; v, t + 1 − T)$ of variables $u$ at time $t$ and $v$ at time $t + 1 − T$,

$$ρ^{(1)}(x, t + 1) = \int_0^1 \int_0^1 \rho^{(2)}(u, t; v, t + 1 − T)$$

$$\times δ(x − f((1 − ε)u + εv)) \, du \, dv,$$  \hspace{1cm} (4.1)

where the constraint arising from the iteration map is accounted for by the Dirac δ-functions. Integrating over the δ-functions, eq. (4.1) can be rewritten as

$$ρ^{(1)}(x, t + 1) = \sum_j \frac{1}{|f'(s_j)|ε}$$

$$\times \int ρ^{(2)}(u, t; \frac{s_j − (1 − ε)u}{ε}, t + 1 − T) \, du,$$  \hspace{1cm} (4.2)

where the sum is extended to the pre-images $s_j = f^{-1}_{(j)}(x)$. Eq. (4.2) is analogous to the Frobenius–Perron equation of 1D maps. Rigorously speaking, a stable solution can be proved to exist, provided that some conditions (typically related to the expansivity of the map and to the existence of a Markov partition) are met [19]. Since it is not easy to generalize such conditions to our case (because of the high dimensionality of the phase-space), we will adopt an empirical point of view, limiting ourselves to verify numerically whether a stationary distribution is eventually reached.

However, the most serious difficulty connected to eq. (4.2) is that it is not a closed equation, requiring the knowledge of $ρ^{(2)}$. Moreover, the
further equation for $\rho^{(2)}$ involves the 3-point probability density. In other words, we are in presence of an open hierarchy of equations which must be closed in some way.

The simplest approximation is the factorization of $\rho^{(2)}$,

$$
\rho^{(2)}(u, t; v, t + 1 - T) \\
= \rho^{(1)}(u, t) \rho^{(1)}(v, t + 1 - T),
$$

(4.3)

whose accuracy will be verified a posteriori, comparing with numerical solutions. According to eq. (4.3), one can write

$$
\rho^{(1)}(x) = \sum_i \frac{1}{|f'(s_i)|} e^{-\frac{(s_j - (1 - \epsilon)u)}{\epsilon}} \times \int \rho^{(1)}(u) \rho^{(1)}(\frac{s_j - (1 - \epsilon)u}{\epsilon}) du,
$$

(4.4)

where we have dropped the time dependence, since we are interested in the stationary distribution only.

The quality of the approximation (4.3) is related to the amplitude of correlations after a time $(T - 1)$. From what we have seen in the previous section, we expect eq. (4.4) to be reasonably correct at least in the small $\epsilon$ region. Notice also the nonlinear character of eq. (4.4) which arises from the closure of the hierarchy of linear equations. A detailed numerical analysis of eq. (4.4) is presented in the next section.

The presence of revival peaks suggests the necessity to take at least partially into account the long-time correlations, which are neglected in (4.3). In the second part of this section, we derive a better approximation scheme, by truncating the hierarchy at the next step. In practice, we write the recursive equation for the joint probability $\rho^{(2)}$ (for the sake of clarity we write from the very beginning the equation for the stationary solution, dropping the dependence on $t$)

$$
\rho^{(2)}(x; y, T - 1) \\
= \int \rho^{(3)}(u; v, T - 1; w, 2(T - 1))
$$

(4.10)

$$
\times \delta(x - f((1 - \epsilon)u + \epsilon v)) \\
\times \delta(y - f((1 - \epsilon)v + \epsilon w)) du dv dw,
$$

(4.5)

which has to considered together with eq. (4.1). After integrating the two $\delta$-functions, we obtain

$$
\rho^{(2)}(x; y, T - 1) = \sum_j \frac{1}{|J_{ij}|}$$

(4.6)

$$
\times \int \rho^{(3)}(u_i(v, x); v, T - 1; w_j(v, y), 2(T - 1)) dv,
$$

where

$$
J_{ji} = \epsilon(1 - \epsilon)f'(s_i(x)) f'(s_j(y)),
$$

(4.7)

with $s_i = f^{-1}(x)$ and $s_j = f^{-1}(y)$. Finally,

$$
u_i(v, x) = \frac{-\epsilon v + s_i}{1 - \epsilon},$$

$$w_j(v, y) = \frac{-(1 - \epsilon)v + s_j}{\epsilon}.
$$

(4.8)

A proper approximation is again needed to close eq. (4.6). We assume the following factorization:

$$
\rho^{(3)}(u; v, T - 1; w, 2(T - 1))
$$

$$
= \rho^{(2)}(u; v, T - 1) \rho^{(2)}(v; w, T - 1)
$$

(4.9)

$$
/ \rho^{(1)}(v),
$$

where the one-dimensional density $\rho^{(1)}$ is obtained from the former eq. (4.2). Eq. (4.9) corresponds approximately to a Markov chain with the signal sampled every $T - 1$ units.

Substitution of relation (4.9) into eq. (4.6) yields the nonlinear functional equation

$$
\rho^{(2)}(x; y, T - 1)
$$

$$
= \sum_j \frac{1}{|J_{ij}|} \int_0^1 \left[ \rho^{(2)}(\frac{s_j - \epsilon w}{1 - \epsilon}; v, T - 1)
$$

(4.10)

$$
\times \rho^{(2)}(\frac{s_j - (1 - \epsilon)u}{\epsilon}, T - 1) \right] \rho^{(1)}(v) dv.
$$

Eqs. (4.4) and (4.10) can, in general, be solved only numerically (this is also the case for the
Bernoulli map). While the results of this analysis are described in the following section, here we briefly discuss the scaling behaviour of the invariant measure close to the extrema of the unit interval. It is well known that the single logistic map at the Ulam point exhibits two inverse square-root divergencies. The simple self-consistent approach developed in the appendix shows that this is no longer true for $x$ approaching 0, where $\rho^{(1)}(x)$ tends to a constant value. Direct estimates of $\rho^{(1)}$ confirm this prediction.

5. Numerical analysis

Here, we present the numerical solution of the approximate equations derived in the previous section. Let us recall that the degree of approximation of eq. (4.4) (first-order truncation) depends crucially on the amplitude of the correlation $C_{T-1}$. In the case of the Bernoulli map, we have seen the presence of a series of revivals. However, such peaks are significantly different from zero only at times exactly multiples of $T$ (i.e. they have a zero width). Therefore, the same simulations suggest that the first-order approximation should be sufficiently accurate.

The numerical solution of eq. (4.4) is determined by fixing an initially flat distribution and generating a new probability density from eq. (4.4) used as a recursive equation. The l.h.s. is interpreted as the new probability density which is again injected in the r.h.s. and the whole procedure is repeated until a reasonable convergence is obtained (in general, a few steps suffice). The integral in the r.h.s. is easily computed by noticing that it can be expressed as a convolution product, an operation effectively performed by exploiting a fast Fourier transform. The stationary distribution for a Bernoulli map is plotted in fig. 10 (dashed line), where it is compared with the histogram generated through a direct simulation. The two curves are practically indistinguishable, thus indicating that factorization (4.3) holds exactly.

![Fig. 10. Invariant measured $\rho^{(1)}(x)$ for the Bernoulli map with $a = b = 2, \varepsilon = 0.20$ (dashed line) compared with a histogram of $3.2 \times 10^6$ iterations over 200 channels ($T = 160$) (solid line). FPE has been solved over a grid of 10000 points, iterating 20 times.]

![Fig. 11. Invariant measure $\rho^{(1)}(x)$ for the asymmetric Bernoulli map with $a_1 = 3.0, a_2 = 1.5$ (dashed line), $\varepsilon = 0.20$ compared with a histogram of $3.2 \times 10^6$ iterations over 200 channels ($T = 160$) (solid line). FPE has been solved over a grid of 20000 points, iterating 20 times.]

In order to better understand the range of validity of approximation (4.3), we have also performed some simulations with the asymmetric Bernoulli shift

$$f(x) = \begin{cases} a_1 x, & x < 1/a_1, \\ a_2 (x - 1)/a_1, & x > 1/a_1. \end{cases}$$

(5.1)

The results plotted in fig. 11 show an excellent agreement, although tiny deviations are now noticeable.

Finally, we have plotted in fig. 12 a probability
distribution obtained for the logistic map. The agreement between direct simulations and the solution of the Frobenius–Perron equation is less pronounced. This is consistent with the behaviour of the correlation functions observed in section 3, which show that the revival peaks extend over more than just one time-lattice point. In this case, it is worth using also the second-order approximation scheme (4.10), which takes partly into account the effect of correlations. The long-dashed curve in fig. 12 shows that a significant improvement is achieved, although the still imperfect agreement indicates the presence of higher-order correlations.

We end this section by discussing various schemes to compute the anomalous Lyapunov exponents. The simplest method is a sort of mean field approach which consists in substituting the delay term in map (2.2) with a constant value \( h \),

\[
y_{r+1} = (1 - \varepsilon)f(y_r) + \varepsilon h,
\]

which is again determined in a self-consistent manner as an average of \( f(y_{r-1} - 1) \), i.e. summing all \( f \)-values corresponding to the arguments \( y_r \) arising from the whole process (5.2). According-

ly, the delayed system is reduced to a 1D map which, in the case of the logistic map (the first nontrivial model where \( f' \) fluctuates), is still a logistic map. The resulting Lyapunov exponent of this effective map is plotted in fig. 1 (solid line). The comparison with the direct simulations shows, apart from the many stability windows, only a qualitative agreement.

A significant improvement is obtained by exploiting eq. (2.6) and performing an ensemble average over the invariant measure of \( y_r \), determined from the Frobenius–Perron equation (see full dots in fig. 1). However, one can also notice that the period-2 stability window is missed by the first-order Frobenius–Perron equation. This is not completely true, since different suitable choices of the initial probability distribution converge correctly to an oscillating density. In other words, the nonlinear eq. (4.4) exhibits a bistable behaviour, and we are left with the ambiguity to decide which is the physical solution.

The analysis carried over these last two Sections has shown that, at variance with some models of class-I, no evidence of a Gaussian-like probability density is found. It is easy to realize that such a discrepancy arises mainly from the nonlinear character of the local coupling. In fact, as discussed in ref. [9], the Gaussian shape of the probability density follows from the interpretation of eq. (1.2) as a linear Langevin equation, with the delay playing the role of a stochastic noise. Here, the presence of a chaotic nonlinear local coupling makes it impossible to generate a Gaussian distribution even when the delayed coupling can be considered as a stochastic noise, as assumed in the previous section.

6. Conclusions

In this paper we have introduced a wider class of dynamical systems with delayed feedback, characterized by a discrete time variable. This choice offers, among other advantages, the
possibility of more accurate simulations on the high-dimensional chaotic phase. Moreover, the local nonlinear interaction is shown to be responsible for the existence of a different component of the Lyapunov spectrum, which does not scale with the delay time. Both analytical and numerical studies reveal a transition where such a component disappears. It is interesting to notice the analogy between this transition and synchronization of chaotic trajectories [20]. In fact, let us recall that eq. (2.6), derived from eq. (2.5) after some approximations, was claimed to yield the exact maximum Lyapunov exponent in the thermodynamic limit \((T \to \infty)\) only if it turns out to be positive. However, we can also reinterpret it as the "exact" equation ruling the linearized sub-dynamics in a suitable 1D space, where the delay is simply seen as a noisy contribution. Accordingly, \(\lambda_{\text{max}}\) is a sub-Lyapunov exponent in the terminology of ref. [20] and its sign reversal can be seen as the transition to a synchronized state with respect to the retarded stochastic action.

\[ p(x \geq x') = \int_A \rho^{(2)}(u; v, T - 1) \, du \, dv. \]  

(A.2)

Under the approximation of a probability density roughly uniform in the set \(A\), we can write

\[ p(x \geq x') \approx \int_A du \, dv \sim \sqrt{1-x'}, \]  

(A.3)

from which the scaling

\[ \rho^{(1)}(x) \sim \frac{1}{\sqrt{1-x}}, \]  

(A.4)

follows, exactly as in the unidimensional case [21].

In the opposite limit \(x = 0\), \(x_{r+1}\) belongs to \([0, x')\) only if \(y_i\) belongs either to the two intervals \(I_1 = [0, \frac{1}{2}x']\) or \(I_2 = [1 - \frac{1}{2}x', 1]\). The probability to have \(x \leq x'\) is then obtained as a sum of two contributions

\[ P(x \leq x') = \int_B \rho^{(2)}(u; v, T - 1) \, du \, dv + \int_C \rho^{(2)}(u; v, T - 1) \, du \, dv, \]  

(A.5)

where the sets \(B\) and \(C\) are defined respectively by the conditions \(0 < (1 - \varepsilon)u + \varepsilon v < \frac{1}{2}x'\) and \(1 - \frac{1}{2}x' < \varepsilon u + (1 - \varepsilon)v \leq 1\). The instability of the map around the origin does not modify the scaling behaviour of the contribution arising from \(C\). Consistently with the previous results, we can assume a divergence like

\[ \rho_2(u; v, T - 1) \sim \frac{1}{\sqrt{1-u} \sqrt{1-v}}. \]  

(A.6)

The computation of the integral over \(C\) in eq. (A.5) then leads to \(p(x \leq x') \sim x',\) that is, \(\rho^{(1)}(x) \sim \text{constant for } x \sim 0\). This result, which is also confirmed by direct numerical simulations, is to be compared with the square-root divergence of the single map. Since \(\varepsilon\) never enters this analysis in determining the scaling behaviour, we can conclude that even a small coupling intro-
duces a qualitative change in the shape of the probability distribution.

References